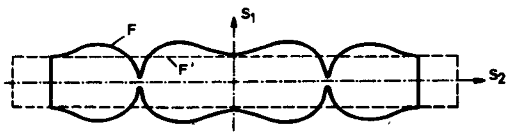

Fig. 10. Characteristic longitudinal section F of the bandpass filter developed.

coupled waveguide filter. At this example, it turned out that it was not possible to influence the eigenvalues of the same order for the open- and short-circuit case independently of one another. Therefore, they were influenced alternately.

The structure shown in Fig. 10 was analyzed by means of a stepped waveguide using formulas given in [6]. The location of the passbands was in very good agreement with the required values.

In the same way as just described, low passes and high passes were synthesized.

IV. Conclusion

A general principle for synthesizing nonuniform waveguides with desired properties was described. The method is an iterative one. The application of the method was described for one kind of nonuniform waveguide with rectangular cross section and excited by a TE_{10} mode. Simple examples have proved the feasibility of the method. Some experience for a successful adaptation of the method were given. In general, the method can be adapted to more complicated problems, e.g., matching problems. A corresponding computer program is under test.

Acknowledgment

The author wishes to thank H. Holl for valuable comments regarding the English translation.

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Short Papers

Tabulation of Methods for the Numerical Solution of the Hollow Waveguide Problem

FOOK LOY NG

***Abstract*—A comparison of methods for the numerical solution of the hollow waveguide problem is presented in tabular form. Another table lists waveguide shapes and their cutoff characteristics that have been presented in the literature. These tables and the bibliography afford an aid towards the selection of a method.**

Introduction

Consider a uniform waveguide with perfectly conducting walls. For the propagation of monochromatic electromagnetic waves inside the waveguide, Maxwell's equations reduce to the two-dimensional Helmholtz equation [1, sect. 8.1].

All analyses of the hollow waveguide problem are attempts at solving, exactly or approximately, the Helmholtz equation subject to the imposed Dirichlet or Neumann boundary conditions for E modes (TM) or H modes (TE), respectively [1, ch. 8].

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Many numerical methods have been proposed and used for the solution of the waveguide problem. A commentary on and comparison of the methods together with relevant references are given in this short paper. A table of the methods and their chief characteristics are presented for convenient reference. Another table is given listing the waveguide shapes that have been treated in the literature. This is provided as a handy reference of shapes that can be used for the testing of any numerical method. This short paper is a condensed version of an earlier publication appearing in a journal with limited circulation [2].

A general introduction to numerical techniques and a review of finite difference and variational techniques for electromagnetic problems are given by Wexler [3]. A review of some current numerical methods for the solution of the waveguide problem is given by Davies [4], and he establishes certain criteria as a basis for comparison of the various methods.

Comparison of Methods

Waveguide shapes can be classified [5] into the three basic types shown in Fig. 1.

In general, type 3 is the most troublesome computationally because of the singular behavior of the field at the reentrant corners [6, sect. 9.2]. Most of the methods either suffer from a slower convergence rate or do not produce reliable results for this type of shape.

The methods that have been used are compared in Table I. Some criteria established by Davies [4] for the comparison of

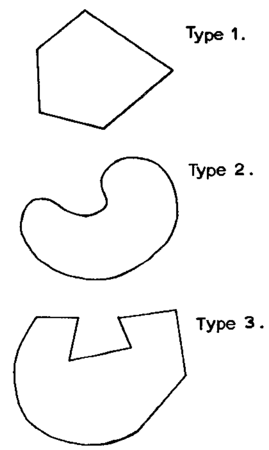


Fig. 1. Classification of waveguide shapes. Type 1—convex. Type 2—nonconvex, with smooth reentrant part(s). Type 3—nonconvex, with reentrant corners.

TABLE I
COMPARISON OF METHODS FOR THE NUMERICAL SOLUTION OF THE HOLLOW WAVEGUIDE PROBLEM[a]

Method	Reference	Cross-sectional shapes (see Fig. 1)	Properties of Matrix/determinant, and orders needed for k accurate to 0.1% for appropriate shapes	Computer program/solution for wavenumber, k	Program Storage Requirements (8-byte words)	CPU time	Asymptotic convergence rate of the error	Remarks
Variational Rayleigh-Ritz	Bulley [23] Bulley and Davies [5]	General, except type 3 cannot be handled for E modes	Dense, Order 30-40	Versatile. Prog. EHPOL mentioned in reference. Standard eigenvalue matrix problem	70 Kbytes	30-70 secs (IBM 360/50)	Not well defined with polynomial order	Types 2 and 3 slower convergence. Good for curved type 1 shapes. Similar method by Thomas [24] and Valenzuela [25].
Finite-element	Silvester [26], [27]. See also [28], [29]	General	Dense, block diagonal. Order 30-100	Versatile. Progs. documented: Silvester [27], Konrad and Silvester [30]. Standard eigenvalue matrix problem	about 200 Kbytes	40 secs (IBM 7094)	M^{-2D}, M = matrix size D = order of polynomial	Types 2 and 3 slower convergence. D = 1 in the simple finite-element method (Silvester [31]).
Finite-difference (PDSOR)	Beaubien and Wexler [32]	General	Sparse, large matrix. Order 5000 - 20000	Versatile. Prog. documented: (Beaubien and Wexler [33]). Eigenvalue matrix problem. Estimate of k required to start iteration	140-220 Kbytes	8 mins/mode (IBM 360/65)	About h^{-1} to h^{-2}, h = mesh size	See also Pontoppidan [34] for an alternative algorithm. An earlier five-point finite-difference scheme (Davies and Muilwyk [22], also [35] gives only the lowest order modes.
Integral operator formulation	Spielman and Harrington [36]. See also [37].	General	Dense, Order 10-30	Versatile, zeros of determinant give values of k, search routine required	Not given (probably around 100 K bytes)	Not given but stated as "substantial"	-	A moment method (Harrington [38]) with triangle functions is used. Field values close to waveguide wall are computed less accurately.
Null field method (NFM)	Bates [39], Ng and Bates [40]. See also [7].	General	Dense, Order 8-12	Versatile, zeros of determinant give values of k, search routine required	70 K bytes	25 secs per evaluation of determinant (10-15 required per mode)	about M^{-12}, M = order of determinant	A representation for the surface current density satisfying the requirements of the corner is needed for a type 3 shape. A representation by pulses gives the CPM.

TABLE I (*Continued*)

Straight-forward point-matching method (SPM)	Yee and Audeh [41],[42]; Bates [39]	Not universally applicable. See [43-46], [8]	Dense, order 8-12	Versatile, zeros of determinant give values of k, search routine required	13 K bytes	1.5 secs per evaluation of determinant (10-15 required per mode)	M^{-15} or better (before levelling off) for appropriate shapes. Oscillates with increasing M when the method is not valid.	Not always valid for Types 2 and 3 shapes. See [47], [7], [8]
Extended point-matching method (EPM)	Ng [7], Bates and Ng, [8].	Extends usefulness of SPM	As for SPM	Depends on detailed shape of C (the rest as for SPM)	16 K bytes	2.3 secs per evaluation (the rest as for SPM)	As for SPM	Produces accurate results for Type 2 and 3 shapes, if suitable representations can be found.
Complete point-matching method (CPM)	Bates [39], Ng. [7].	General	As for SPM	As for SPM	As for SPM	As for SPM	As for SPM, Generally oscillates for increasing M for types 2 and 3 shapes.	Errors may be large (10-20%) for types 2 and 3 shapes.
Conformal transformation	Meinke & Baier [48], Meinke et al. [49]. See also [50-52].	Best suited to shapes for which simple transformation functions can be found	Dense, order 80-90 for accuracies to 1%	Versatile. Eigenvalue matrix problem.	-	12 mins/mode (TR4) for a complete calculation (including the field)	-	For shapes requiring complicated mapping functions or for higher order modes, the large number of terms which have to be taken into account causes a fall in accuracy.
Perturbation (geometrical approximation)	Uptain & Audeh [53], Pyle and Angley [54], Hu and Ishimaru [55]. See also [1], [illegible] [56].	Limited shapes	Dense	Each waveguide becomes a separate problem. See individual references.	-	-	-	See also Pyle [11] for transverse resonance method and Collins & Daly [12] and Veselov [13] for partial regions method.
Coupled first order operators	Harrington [38] sec. 8.5] (see also Davies [4])	General	Dense	Versatile, zeros of determinant give values of k	-	-	-	Convergence should be better than a second order differential operator method but the matrix size will be larger (Davies [4]).
Transmission-line matrix method	Johns [57]	General	Matrix order of 10-50 required to store node values.	Versatile, solution by iteration of transmission-line matrix equations.	25 kbytes	Fairly excessive as 200-500 iterations are required	-	Method is equivalent to transverse resonance method but divides the waveguide into many rectangular transmission line sections. Curved boundaries require smaller meshes and more storage. Errors of 0.2-0.5% typical.

[a] The properties of the matrix (column 4) and CPU time (column 7) quoted are those required to give results accurate to 0.1 percent in general, unless otherwise stated. All the methods, except for an earlier finite-difference scheme, are capable of predicting the higher order modes in addition to the first E and H modes.

methods are incorporated into the table. The storage requirements quoted are for programs with 8-byte (64-bit) words.

There is, of course, no best method, but the variational, finite element, finite difference, integral operator, null field, conformal transformation, and transmission-line matrix methods are of wide applicability and can be used with all three types of shapes, with suitable modifications where necessary for a type-3 shape.

Point matching methods are attractively economic from the points of view of programming effort and computer time, but they often lose their effectiveness with complicated shapes [7], [8]. The extended point matching method can produce accurate results, however, for shapes that are strongly nonconvex [8].

The perturbation technique and other methods not listed in Table I, including the network impedance analog [9], transverse resonance [10], [11], and partial regions [12], [13] methods, are not of general applicability and lend themselves to particular classes of shapes only. Another technique, the Monte Carlo method, can be used to solve the Helmholtz equation [14], [15], but the method requires excessive computing time in the simulation of a sufficient number of random walks. Its attractive feature is the small computer storage required.

Analog/hybrid computation techniques may be employed, although these methods are still new [3]. A discussion of analog techniques for partial differential equations is given by Fifer [16].

Mention should be made of the classical method of separation of variables (an analytical method), which can be used when the cross section of a waveguide coincides with coordinate surfaces of a separable coordinate system [17]-[20].

No comprehensive comparison of the methods on a single computer is available at present, and different machines have been used in the various methods listed in Table I. A useful comparison of the characteristics of digital computers can be found in [21].

TABLE II

WAVEGUIDE SHAPES

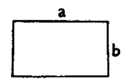
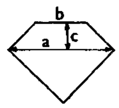
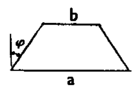
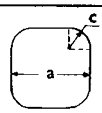
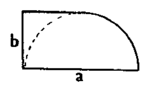
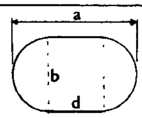
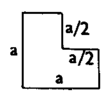
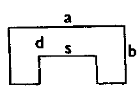
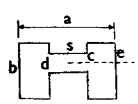
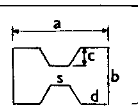
Shape		Dimensions (cutoff wavenumber ka, of lowest order mode given in reference)	H mode	E-mode	Reference	Method used in Reference
Rectangle		b = a/2	3.1416	7.0248	Standard shape e.g. [1], [58]	Separation of Variables
Truncated Square		b=0.55a, c=0.225a	4.215		Bulley and Davies [5]	Variational Rayleigh-Ritz
Trapezoid		b/a=0.25, $\varphi = 0^o(10^o)60^o$. Table of ka values			Uptain and Audeh [53]. See also Audeh & Fuller [59], Veselov & Platonov [13], Chopra & Durvasula [60], [61]	Transverse resonance
Square with rounded corner		c/a = 0 to 1/2 Graph of ka values			Lagasse and Van Bladel [62]	Finite element
"Rounded" Rectangle		No ka listed. Field plots given.			Bulley [23]	Variational Rayleigh-Ritz
Rectangle with Semi-circular sides		b/a = 0 to 1.0. H-mode. Graph of ka values			Valenzuela [23] See also [62]	Variational Rayleigh-Ritz
L-shape				4.819	Reid and Walsh [63]. See also Davies and Nagenthiram [47]	Finite-difference
Single Ridge		b/a=s/a=d/b=1/2.	2.2566	12.164	Spielman and Harrington [36].	Integral Operator
			2.412	12.1416	Beaubien and Wexler [32].	Finite-difference PDSOR
			2.2627		Bulley and Davies [5].	Variational Rayleigh-Ritz
			2.250	12.134	Ng [7], Bates and Ng [8]	Extended point-matching
		b/a = 0.45; d/b,s/a=0.05(0.05) 0.95. Table of ka values.			Pyle [11].	Transverse resonance
Ridge, with unequal ridge depths		a = 0.5, b = 0.4, s = 0.1, c = 0.055, e = 0.02, 0.13, 0.17; d/b = 0.225 to 0.325. H-mode. Graph of ka values.			Montgomery [64]. See also Beaubien and Wexler [65]	Ritz-Galerkin
Rectangle, with trapezoidal ridges		b/a = 0.75, s/a = d/a = 0.15, c/a = 0.2	2.53	7.61	Meinke et al. [49].	Conformal transformation

TABLE OF WAVEGUIDE SHAPES

Many waveguide shapes are used in the literature as examples for solutions of the waveguide problem. These range from the simple trapezoid to the exotic club shape of Davies and Muilwyk [22]. Recently, attention has been focused on the challenging and practical ridge waveguide (a type-3 shape).

Because of the profusion of shapes that have been used at one time or another it is felt that Table II, which lists these shapes, will prove useful for anyone wishing to check a new method. Also, the

TABLE II
(Continued)

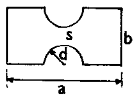
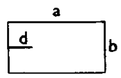
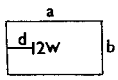
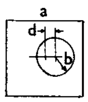
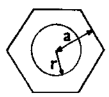
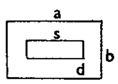
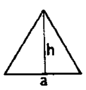
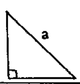
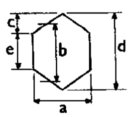
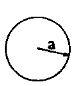
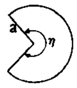
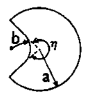
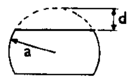
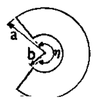
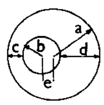
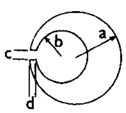
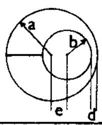
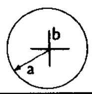
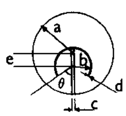
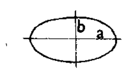
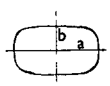
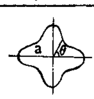
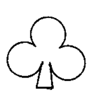
Meinke guide		a/b=1.29, d/b=0.3			
		2.267		Bulley and Davies [5]	Variational Rayleigh-Ritz
		2.268		Ng [7], Bates and Ng [8]	Extended point-matching
		b/a = 0.775, s/a = 0 to 1.0 Graph of ka values		Meinke and Baier [48].	Conformal transformation
Vaned rectangle		b/a = 0.25 (0.25) 1.0, d/a = 0 to 0.8. Graph of ka values		Silvester [26]	Finite-element
T-septate rectangle		b/a = 4W, d/a = 0.26, W/a = 0.311/2.744		Beaubien and Wexler [32]. See also Silvester [26].	Finite-difference PDSOR
		2.9682	8.1181		
Square outer, eccentric circular inner		2a = 0.125, 0.250, 0.375 d/(a-b) = 0 to 1.0 Graphs of ka values		Audeh and Fuller [59].	(Extended) point-matching
n-sided polygonal outer, co-axial circular inner.		n = 4,5,6,7 and 8 r/a = 0 to 1.0 Lowest order E mode. Graphs of ka values		Laura et al. [50]	Conformal transformation
Coaxial rectangles		b/a = 0.8, 2d/b = 0.6 s/a = 0 to 1.0 Graph of ka values		Gruner [66].	Partial regions
Equilateral triangle		4π/3	2πa/h	Schelkunoff [17], sec. 10.8]. See also Thomas [24].	Analytic solution
Isosceles right-angled triangle		4.43	9.96	Schelkunoff [17], sec. 10.8] and Morse and Feshbach [67, p.756].	Analytic solution
Regular pentagon			2.285	Laura [68]	Conformal transformation
Regular hexagon			2.317	" "	" "
Regular heptagon			2.339	" "	" "
Regular octagon			2.355	" "	" "
Hexagon		b/a = 0.9, d/b = 1.0 to 1.15 Graph of ka values		Meinke and Baier [48]	Conformal transformation
		c/a = 0 to 5.0, for several values of cross-sectional area. Graph of ka values.		Schlosser [69]	Transverse resonance
Star shaped		Dimensions not given, see figure in reference		Thomas [24]	Variational Galerkin's
			k = 4.087		
Circle		1.8412	2.4048	Standard shape e.g. [1].	Separation of variables

TABLE II
(Continued)

Sector		$\eta = \pi/2$ 3.0542 5.1356 $\eta = 3\pi/2$ 1.4012 3.3756	Ng [7], see also Ng and Bates [40].	Separation of variables
"Rounded" sector		$\eta = 3\pi/2$, $b/a = 0.35$ 3.2015	Bates and Ng [8].	Null field method
Truncated circle		d/a = 0 to 0.3 E mode. Graph of ka values.	Pyle and Angley [54]. See also Sinnott [70] and Schlosser [69].	Perturbation and transverse resonance
Segment		$\eta = \pi/2$, b/a = 0.2 5.2218 $\eta = 3\pi/2$, b/a = 0.3 4.5457	Ng [7], Bates and Ng [8].	Separation of variables
Circular outer, eccentric circular inner		b/a = 0.25, 0.5; e/b = 0 to 3.0 Graphs of ka values	Yee and Audeh [42].	(Extended) point-matching
		b/a = 0.434, c/d = 0.1 to 1.0 Graphs of ka values	Abaka and Baier [52]. See also Veselov and Semenov [71], and Dwight [72].	Conformal transformation
Lunar shape		a = 13, b = 7.435, c = 1.0, d = 1.43 0.990 4.605	Beaubien and Wexler [32]. See also Meinke and Baier [48], Hu and Ishimaru [55], Arlett et al. [28]	Finite-difference PDSOR
Inverted lunar		a = 34, d = 3.74, e = 38.9 0.770	Meinke et al. [49]. See also Meinke and Baier [48].	Conformal transformation
Circle, with central cross		b/a = 0 to 1.0 Graph of ka values	Veselov and Gaydar [73].	Partial regions
T-septate circle		a = 13, b = 6.875, c = 1.125, d = 0.5, e = 3.375, $\theta = 22.5°$ 0.517 4.903	Beaubien and Wexler [32]. See also Hu Wang [74], Arlett et al. [28].	Finite-difference PDOSR
Ellipse		$(x/a)^2 + (y/b)^2 = 1$, $e^2 = 1 - b^2/a^2$. e = 0.0 to 1.0. E and H modes. Graphs of ka values	Krestzschmar [18], See also Rayevskiy and Smorgonskiy [75], Davies and Kretzschmar [56], Larsen [58].	Separation of variables
Super-ellipse		$(x/a)^n + (y/b)^n = 1$ b/a = 0.3 to 1.0, n = 2 to ∞. H-mode. Graph of ka values.	Larsen [58], [76].	Finite-difference
Parabolic		see reference	Horiuchi et al. [20] and Zagrodzinski [19]. Also [58].	Separation of variables
"Star" shape		$\rho = 1 + b \cos 4\theta$, b = 0 to 0.3 H mode. Graph of ka values	Laura [77].	Conformal transformation
Club shape		See reference 3.32	Davies and Muilwyk [22].	Finite-difference

information contained in Table II is fairly comprehensive and is worth presenting for its own sake.

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Numerical Solution of Surface Waveguide Modes Using Transverse Field Components

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Abstract—**The computation of surface waveguide modes is facilitated by reducing the surface waveguide field problem to a conventional eigenvalue problem that has no spurious solutions. This is achieved by formulating the field problem in terms of transverse field components and by using impedance boundary conditions on an auxiliary boundary with a specified value of the exterior cutoff wavenumber.**

Introduction

In many field problems of practical interest, the region being considered is of infinite extent. A numerical method [1]–[3] which combines integral and differential equation approaches is found to be effective in increasing computational efficiency and accuracy. A further application of the method is described here, namely, the computation of surface waveguide modes. When formulated in terms of transverse field components, this is a two-dimensional exterior eigenvalue problem.

Selection of Field Components

A surface waveguide is essentially an inhomogeneous waveguide without a closed boundary. The wave equation describing the propagation in an inhomogeneous waveguide can be expressed in terms of two field components, which are usually taken to be the longitudinal components, E_z and H_z. (A field dependence of $\exp[j(\omega t - \beta z)]$ is assumed throughout.) However, as pointed out by Gelder [4], this choice leads to a generalized eigenvalue problem which, for a specified angular frequency ω, is nonlinear in the eigenvalue β^2. If the phase velocity ω/β is specified instead, a conventional problem with eigenvalue ω^2 is obtained, but the solutions include spurious nonsurface modes. This is because the exterior field of a surface mode decays exponentially corresponding to an imaginary exterior cutoff wavenumber k_A, that is, $k_A^2 = k_0^2 - \beta^2 = \omega^2\mu_0\epsilon_0 - \beta^2$ is negative for a surface mode, whereas the specification of ω/β is insufficient to determine k_A^2. On the other hand, for a specified value of k_A^2, use of the transverse components [4], E_x and E_y, or H_x and H_y, leads to a conventional eigenvalue problem with eigenvalue ω^2 which has no spurious solutions.

Problem Formulation

The cross section of a typical surface waveguide is shown in Fig. 1. The rectangular dielectric rod (permittivity ϵ) is enclosed within an auxiliary boundary C which divides all space into an interior region R

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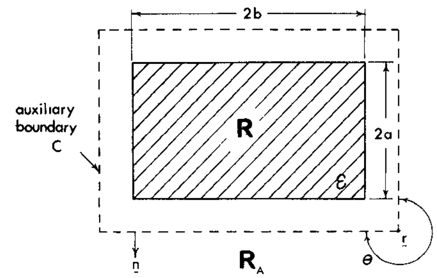

Fig. 1. Cross section of a rectangular dielectric rod surface waveguide.

and a homogeneous exterior region R_A. The transverse magnetic field satisfies the differential equation [5]

$$\nabla_t\left[\frac{1}{\mu}\nabla_t\cdot(\mu \boldsymbol{H}_t)\right] - \epsilon\nabla_t \times \left[\frac{1}{\epsilon}(\nabla_t \times \boldsymbol{H}_t)\right] = (\beta^2 - \omega^2\mu\epsilon)\boldsymbol{H}_t. \quad (1)$$

Assuming uniform permeability μ_0, it is convenient to rearrange (1) into the following component form:

$$-(\nabla_t^2 + k_A^2)H_x - \frac{1}{\epsilon}\frac{\partial\epsilon}{\partial y}\left(\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x}\right) = \omega^2\mu_0(\epsilon - \epsilon_0)H_x \quad (2)$$

$$-(\nabla_t^2 + k_A^2)H_y = \frac{1}{\epsilon}\frac{\partial\epsilon}{\partial x}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = \omega^2\mu_0(\epsilon - \epsilon_0)H_y \quad (3)$$

which reduces to

$$-(\nabla_t^2 + k_A^2)\boldsymbol{H}_t = 0 \quad (4)$$

in the homogeneous exterior region R_A. Although (1) is not self-adjoint, it can be solved in R by such conventional techniques as the method of moments [6]. For example, projecting both sides of (2) and (3) onto the space spanned by a set of testing functions $W_i(x,y)$ yields

$$\iint_R \left\{\frac{\partial W_i}{\partial x}\frac{\partial H_x}{\partial x} + \frac{\partial W_i}{\partial y}\frac{\partial H_x}{\partial y} - k_A^2 W_i H_x - \frac{1}{\epsilon}\frac{\partial\epsilon}{\partial y}W_i\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\right\} dA - \oint_C W_i \frac{\partial H_x}{\partial n}\, ds = \omega^2 \iint_R \mu_0(\epsilon - \epsilon_0) W_i H_x\, dA \quad (5)$$

$$\iint_R \left\{\frac{\partial W_i}{\partial x}\frac{\partial H_y}{\partial x} + \frac{\partial W_i}{\partial y}\frac{\partial H_y}{\partial y} - k_A^2 W_i H_y - \frac{1}{\epsilon}\frac{\partial\epsilon}{\partial x}W_i\left(\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x}\right)\right\} dA - \oint_C W_i \frac{\partial H_y}{\partial n}\, ds = \omega^2 \iint_R \mu_0(\epsilon - \epsilon_0) W_i H_y\, dA \quad (6)$$

where n is the outward normal. In addition, the transverse field components must also satisfy (4) in the homogeneous exterior region R_A. Hence the trial values of the transverse field $\boldsymbol{H}_{tC}$ and its outward derivative $\partial\boldsymbol{H}_{tC}/\partial n$ on the auxiliary boundary cannot be independent. The compatibility condition which links them is found by applying Green's theorem to (4) to yield the integral equation

$$\boldsymbol{H}_{tC}(\boldsymbol{r}) = \frac{1}{\theta}\oint_C \left\{\boldsymbol{H}_{tC}(\boldsymbol{r}_0)\frac{\partial}{\partial n}K_0(k\,|\,\boldsymbol{r} - \boldsymbol{r}_0\,|) - K_0(k\,|\,\boldsymbol{r} - \boldsymbol{r}_0\,|)\frac{\partial \boldsymbol{H}_{tC}}{\partial n}(\boldsymbol{r}_0)\right\} ds_0 \quad (7)$$

where $k = (-k_A^2)^{1/2}$, $K_0(k\,|\,\boldsymbol{r} - \boldsymbol{r}_0\,|)$ is a modified Bessel function [Green's function for (4)], θ is the exterior angle in radians between the tangents on each side of the point $\boldsymbol{r}$ on C, and it is understood that